The Vasicek Interest Rate Process
Options On Bonds

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We are currently standing at time n and want to determine the price to be paid at time n for a call option on a zero coupon, default free bond that pays one dollar at time t and can be purchased at time s where $n < s < t$. In this white paper we will build a model to value that option. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are currently standing at time zero and are tasked with determining the price to be paid at time zero for an option on a risk-free zero coupon bond. Our go-forward model assumptions are...

Description	Symbol	Value
Bond face value (in dollars)	FV	1,000
Option exercise price (in dollars)	K	725
Option term (in years)	s	3.00
Bond term (in years)	t	7.00
Short rate at time zero	r_0	0.04
Long-term short rate mean	r_∞	0.09
Short rate volatility	σ	0.03
Mean reversion rate	λ	0.35

Question 1: What is the price to be paid at time zero for a call option on this bond?

Question 2: What is random bond price at the end of the option term under the risk-neutral probability distribution given that the random number 0.85 was pulled from that distribution?

Bond Price Equations

The asset underlying the call option in our hypothetical problem above is a zero coupon bond that can be purchased at time s and pays one dollar at time t. We will define the variable $P(s,t)$ to be the price to be paid at time s for a zero coupon bond that pays one dollar at time t, and the variable $r_{s,s}$ to be the short-rate at time s from the standpoint of time s. The equation for bond price at time s from the standpoint of time s is... [3]

$$P(s,t) = \mathrm{Exp}\left\{A(s,t) - B(s,t)\, r_{s,s}\right\} \text{ ...where... } r_{s,s} \text{ is the known short rate at time } s \text{ ...and...}$$

$$A(s,t) = \left(r_\infty - \frac{\sigma^2}{2\,\lambda^2}\right)\left(B(s,t) - (t-s)\right) - \frac{\sigma^2}{4\,\lambda}\,B^2(s,t) \text{ ...and... } B(s,t) = \left(1 - \mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\right)\lambda^{-1} \tag{1}$$

Note that in Equation (1) above the short rate $r_{s,s}$ is known at time s and therefore the price of the bond at time s is not a random variable.

To price our option we will need the risk-free rate over the option exercise time interval $[n,s]$. We will define the variable $P(n,s)$ to be the price to be paid at time n for a zero coupon bond that pays one dollar at time s, and the variable $r_{n,n}$ to be the short-rate at time n from the standpoint of time n. Using Equation (1) above the equation for bond price at time n from the standpoint of time n is...

$$P(n,s)=\text{Exp}\left\{A(n,s)-B(n,s)\,r_{n,n}\right\} \tag{2}$$

Note that in Equation (2) above the short rate $r_{n,n}$ is known at time n and therefore the price of the bond at time n is not a random variable.

We will define the variable α to be the annualized, continuous-time risk-free rate over the time interval $[n,s]$. Noting that the bond in Equation (2) above is risk-free (i.e. default-free) then using that bond price the equation for the risk-free rate is...

$$\text{if... } P(n,s)=\$1.00\times\text{Exp}\left\{-\alpha\,(s-n)\right\}\ \text{...then... } \alpha=-\frac{\ln P(n,s)}{s-n} \tag{3}$$

Given that we are currently standing at time n, we need an equation for the expected price of bond $P(s,t)$ (our call option's underlying asset) in Equation (1) above at time s from the perspective of time n. We will define the variable $F(n,s,t)$ to be the forward price to be paid at time s for a zero coupon bond that pays one dollar at time t from the standpoint of time n. Using Equation (1) above the equation for forward bond price at time s from the standpoint of time n is... [4]

$$F(n,s,t)=\frac{P(n,t)}{P(n,s)} \tag{4}$$

Note that in Equation (4) above the forward price of the bond $P(s,t)$ is determined at time n but the actual price of the bond at time s is not known at time n and is therefore a random variable.

We will define the variables M_s to be the **expected** market value of the option's underlying asset at future time s and the variable M_n to be the **known** market value of that asset at current time n. Using Equations (2), (3) and (4) above the equations for the asset underlying our option are...

$$\mathbb{E}\left[M_s\right]=FV\times F(n,s,t)\ \text{...and... } M_n=\mathbb{E}\left[M_s\right]\times P(n,s)=FV\times F(n,s,t)\times\text{Exp}\left\{-\alpha\,(s-n)\right\} \tag{5}$$

We will define the variable υ to be the random bond price annual variance. Using Equation (5) above the equation for random bond price at time s as a function of known bond price at time n is... [1]

$$M_s=M_n\,\text{Exp}\left\{\left(\alpha-\frac{1}{2}\,\upsilon\right)(s-n)+\sqrt{\upsilon\,(s-n)}\,Z\right\}\ \text{...where... } Z\sim N\left[0,1\right] \tag{6}$$

Using Equation (1) above the equation for the annual variance of random bond return over the time interval $[n,s]$ in Equation (6) above is... [4]

$$\upsilon=\left[B(s,t)^2\,\frac{1}{2}\,\sigma^2\left(1-\text{Exp}\left\{-2\,\lambda\,(s-n)\right\}\right)\lambda^{-1}\right]\Big/(s-n) \tag{7}$$

Note that the variable σ in Equation (7) above is the volatility of the short rate.

The Call Option Model

If bond price M_s is greater than the exercise price K at time s then the option is in the money and will be exercised such that the option holder realizes a gain of M_s-K. If bond price M_s is less than the exercise price K at time s then the option is out of the money and will not be exercised. We will define the variable C_n to be option value at time n. The equation for option value at time n is... [2]

$$C_n=\mathbb{E}^Q\left[\text{Max}\left\{M_s-K,0\right\}\text{Exp}\left\{-\alpha\,(s-n)\right\}\right] \tag{8}$$

Note that we can rewrite option price Equation (8) above as...

$$C_n=\int\limits_a^\infty\sqrt{\frac{1}{2\,\pi}}\,\text{Exp}\left\{-\frac{1}{2}\,Z^2\right\}\left[M_n\,\text{Exp}\left\{\left(\alpha-\frac{1}{2}\,\upsilon\right)(s-n)+\sqrt{\upsilon\,(s-n)}\,Z\right\}-K\right]\text{Exp}\left\{-\alpha\,(s-n)\right\}\delta Z$$

$$\text{where... } a=\left[\ln\left(\frac{K}{M_n}\right)-\left(\alpha-\frac{1}{2}\,\upsilon\right)(s-n)\right]\Big/\sqrt{\upsilon\,(s-n)} \tag{9}$$

We will define the function $CNDF$ to be the cumulative normal distribution function of a normally-distributed random variable with mean zero and variance one. The solution to the integral in Equation (9) above is...

$$C_n = M_n \, \text{CNDF}\left[d_1\right] - K \, \text{Exp}\left\{ - \alpha \, (s-n)\right\} \text{CNDF}\left[d_2\right]$$
$$\text{where... } d_1 = -a + \sqrt{\upsilon \, (s-n)} \text{ ...and... } d_2 = -a \tag{10}$$

Note that Equation (10) above is the Black-Scholes Option Pricing Model.

The Answers To Our Hypothetical Problem

Using Equation (1) above the equation for the bond pricing parameter B is...

$$B(0,3) = \left(1 - \text{Exp}\left\{ - 0.35 \times (3-0)\right\}\right) \times 0.35^{-1} = 1.85732$$
$$B(0,7) = \left(1 - \text{Exp}\left\{ - 0.35 \times (7-0)\right\}\right) \times 0.35^{-1} = 2.61059$$
$$B(3,7) = \left(1 - \text{Exp}\left\{ - 0.35 \times (7-3)\right\}\right) \times 0.35^{-1} = 2.15258 \tag{11}$$

Using Equations (1) and (11) above the equation for the bond pricing parameter A is...

$$A(0,3) = \left(0.09 - \frac{0.03^2}{2 \times 0.35^2}\right) \times \left(1.85732 - (3-0)\right) - \frac{0.03^2}{4 \times 0.35} \times 1.85732^2 = -0.10086$$
$$A(0,7) = \left(0.09 - \frac{0.03^2}{2 \times 0.35^2}\right) \times \left(2.61059 - (7-0)\right) - \frac{0.03^2}{4 \times 0.35} \times 2.61059^2 = -0.38330 \tag{12}$$

Using Equations (2), (11) and (12) above the equation for the risk-free bond price is...

$$P(0,3) = \text{Exp}\left\{ - 0.10086 - 1.85732 \times 0.04\right\} = 0.83933$$
$$P(0,7) = \text{Exp}\left\{ - 0.38330 - 2.61059 \times 0.04\right\} = 0.61402 \tag{13}$$

Using Equations (3) and (13) above the risk-free rate over the time interval $[0,3]$ is...

$$\alpha = -\frac{\ln P(n,s)}{s-n} = -\frac{\ln 0.83933}{3-0} = 0.0584 \tag{14}$$

Using Equations (4) and (13) above the equation for forward bond price is...

$$F(0,3,7) = \frac{P(0,7)}{P(0,3)} = \frac{0.61402}{0.83933} = 0.73156 \tag{15}$$

Using Equations (5), (14) and (15) above the equation for bond price at time zero (the value of the asset underlying the option) is...

$$M_0 = 1{,}000.00 \times 0.73156 \times \text{Exp}\left\{ - 0.0584 \times (3-0)\right\} = 614.02 \tag{16}$$

Using Equations (7) and (11) the equation for the annual variance of the log of random bond price is...

$$\upsilon = \left(2.15258^2 \times \frac{1}{2} \times 0.03^2 \times \left(1 - \text{Exp}\left\{ - 2 \times 0.35 \times (3-0)\right\}\right) \times 0.35^{-1}\right) \Big/ (3-0) = 0.00174 \tag{17}$$

Using Equations (9), (14), (16), and (17) above the equation for option model variable a is...

$$a = \left[\ln\left(\frac{725.00}{614.02}\right) - \left(0.0584 - \frac{1}{2} \times 0.00174\right) \times (3-0)\right] \Big/ \sqrt{0.00174 \times (3-0)} = -0.08846 \tag{18}$$

Using Equations (10), (17) and (18) above the equations for option parameters d_1 and d_2 are...

$$d_1=-a+\sqrt{0.00174\times(3-0)}=0.16077 \text{ ...and... } d_2=-a=0.08846 \tag{19}$$

The cumulative normal distribution function of the parameters d_1 and d_2 in Equation (19) above are...

$$CNDF\left[d_1\right]=0.56386 \text{ ...and... } CNDF\left[d_2\right]=0.53525 \tag{20}$$

Question 1: What is the price to be paid at time zero for a call option on this bond?

Using Equations (10), (15), (16) and (20) above the answer to the question is...

$$C_0=614.02\times 0.56386-725.00\times \mathrm{Exp}\left\{-0.0584\times(3-0)\right\}\times 0.53525=20.52 \tag{21}$$

Question 2: What is random bond price at the end of the option term under the risk-neutral probability distribution given that the random number 0.85 was pulled from that distribution?

Using Equations (6), (14), (16) and (17) above the answer to the question is...

$$M_3=614.02\times \mathrm{Exp}\left\{\left(0.0584-\frac{1}{2}\times 0.00174\right)\times(3-0)+\sqrt{0.00174\times(3-0)}\times 0.85\right\}=775.90 \tag{22}$$

References

[1] Gary Schurman, *Brownian Motion - An Introduction To Stochastic Calculus,* February, 2012.

[2] Gary Schurman, *Deriving The Black-Scholes Model Via Risk-Neutral Probabilities,* October, 2010.

[3] Gary Schurman, *The Vasicek Interest Rate Process - An Alternative Bond Price Equation,* December, 2021.

[4] Gary Schurman, *The Vasicek Interest Rate Process - Modeling Random Bond Price,* March, 2022.